Many-particle wave functions

Once one learns to solve one-body quantum mechanics problems, one needs to understand how to solve systems with more particles.

You must have done this when you started with quantum field theory, but let me remind you.

~~A single particle in 3 dim. is described by a wavefn.~~ ~~$\psi(\vec{x})$~~

A single spin $\frac{1}{2}$ has a Hilbert space of $d = 2$, because each spin can be $\uparrow$ or $\downarrow$.

If we have 2 spins, then the Hilbert space has $d = 2 \times 2$

|||ly if we have N spins, the Hilbert space $= d^N$.

|||ly, if we have a single ptcle in 3 dimensions with wfn $\psi(\vec{x})$, then the Hilbert space is ∞

but if it is made finite,
then the dimension can be counted
as d.
But then if we have ~~2~~ 2
particles, the Hilbert space,
is $d \times d$.
And if we have N ptcles,
then the Hilbert space is d^N.

If there particles are all
interacting with each, then it is
an extremely difficult problem, because
the Hilbert space is so big
that even trying to diagonalise it
on any computer is not practically
possible. Even with spin models
with Hilbert space of 2 (or 3 or 4),
one can diagonalise only small systems
otherwise one needs to use approximations.
This is why interacting many ptcle
systems — cond-mat systems — are hard
to study.
But at least here, we will not
consider interactions.

Here, all I want is to mention non-interacting systems and generalise it to one very specific interacting system.

In quantum mechanics, we assume that ptcles are indistinguishable - i.e. ptcles with all other quantum #'s such as spin being the same will be indistinguishable. So if we exchange 2 such ptcles, the wave-fn will be the same as before

$$\psi(\vec{q_2}, \vec{q_1}) = \psi(\vec{q_1}, \vec{q_2})$$

In pple. Is this right? No. ~~upto~~, only you could get a phase, since the mod squared of the wfn is physical.

$$\text{So} \quad \psi(\vec{q_2}, \vec{q_1}) = e^{i\varphi} \psi(\vec{q_1}, \vec{q_2})$$

But after 2 exchanges you have to come back to the original wave fn

$$\psi(\vec{q_1}, \vec{q_2}) = e^{2i\varphi} \psi(\vec{q_1}, \vec{q_2})$$

$$\Rightarrow e^{2i\varphi} = 1 \Rightarrow \psi(\vec{q_2}, \vec{q_1}) = \pm \psi(\vec{q_1}, \vec{q_2})$$

So we can have either fermions or bosons - I will come back to this point later.

So, now we need to use either symmetrised or anti-symmetrised wave-fns:

Bosons : $\psi^{S}(\vec{q_1}, \vec{q_2}) = \frac{1}{\sqrt{2}}\left(\psi(\vec{q_1}, \vec{q_2}) + \psi(\vec{q_2}, \vec{q_1})\right)$

Fermions : $\psi^{A}(\vec{q_1}, \vec{q_2}) = \frac{1}{\sqrt{2}}\left(\psi(\vec{q_1}, \vec{q_2}) - \psi(\vec{q_2}, \vec{q_1})\right)$

$\rightarrow$ normalisation

Now for an N - ptcle system

$$\psi^{S}(\vec{q_1} \ldots \vec{q_N}) = N_S \sum_P \psi(\vec{q}_{P(1)} \ldots \vec{q}_{P(N)})$$

where the sum is over all possible permutations of the N ptcles for bosons.

For fermions, of course, we also have the spin quantum #. But for fermions with the same spin, we need to anti-symmetrise the spatial part, because the spin part is symmetric.

Then $\psi^{A}(\vec{q_1} \ldots \vec{q_N}) = N_S \sum_P (-1)^P \psi(\vec{q}_{P(1)} \ldots \vec{q}_{P(N)})$

There is a convenient way to write this called the Slater determinant

For 2 particles, we know that

$$\cancel{\psi^S(\vec{q_1},\vec{q_2}) = \frac{1}{\sqrt{2}} \left(\psi(q_1)\,\psi(q_2 \right.}$$

particle 2 at position 1

$$\psi^S(\vec{q_1},\vec{q_2}) = \frac{1}{\sqrt{2}} \left[\psi(\vec{q_1},\vec{q_2}) - \psi(\vec{q_2},\vec{q_1}) \right]$$

↳ Ptcle 1 at position 1

For non-interacting ptcles, we can write the 2 particle wave-fn as a product of 2 one-particle wfns.

$$\cancel{\text{So } \psi^S(\vec{q_1},\vec{q_2}) = \frac{1}{\sqrt{2}} \left(\psi(\vec{q_1})\,\psi(\vec{q_2}) - \psi(\vec{q_2})\,\psi(\vec{q_1}) \right)}$$

$$\cancel{= \frac{1}{\sqrt{2}} \Big| \; \psi(q_1)}$$

So

$$\psi^A(\vec{q_1},\vec{q_2}) = \frac{1}{\sqrt{2}} \left[\phi_1(\vec{q_1})\,\phi_2(\vec{q_2}) - \phi_2(\vec{q_1})\,\phi_1(\vec{q_2}) \right]$$

$$= \frac{1}{\sqrt{2}} \begin{vmatrix} \phi_1(\vec{q_1}) & \phi_2(\vec{q_1}) \\ \phi_1(\vec{q_2}) & \phi_2(\vec{q_2}) \end{vmatrix} \longrightarrow \text{det.}$$

Generalising this to many particles, we get the Slater determinant.

$$\psi^A(\vec{q_1}, \ldots \vec{q_N}) = N \begin{vmatrix} \phi_1(\vec{q_1}) & \phi_2(\vec{q_1}) & \cdots & \phi_N(\vec{q_1}) \\ \phi_1(\vec{q_2}) & \phi_2(\vec{q_2}) & \cdots & \phi_N(\vec{q_2}) \\ \vdots & & & \\ \phi_1(\vec{q_N}) & & & \phi_N(\vec{q_N}) \end{vmatrix}$$

Now, let us apply this to the problem of the quantum Hall effect. Let us work in the symmetric gauge, so that I can use $z, \bar{z}$ notation. Then I said that in the lowest LL, we have only analytic functions. So let us try to write down the Slater determinant wave-function of 1 filled Landau level. We want to write down the wave fn of N-particles $\psi(z_1, z_2 \ldots z_N)$ from the single ptcle wave-fns which are given by

$$\psi_0 = 1 \cdot e^{-|z|^2/2} \quad \psi_1 = z\, e^{-|z|^2/2}, \quad \psi_2 = z^2 e^{-|z|^2} \quad \ldots\ldots$$

All these states are obtained by applying $(b^+)|0\rangle$

$\left(b^+ = \frac{-\partial + z}{\sqrt{2}}\right.$ — i.e. it increases factors of z)

So $|0\rangle$, $b^+|0\rangle$, $(b^+)^2|0\rangle$ --- are all states in the lowest Landau level.

So I can write down an anti-symmetric wfn of 2 prtcles as

$$\psi(z_1, z_2) = \cancel{\begin{vmatrix} 1 & 1 \\ z_1 & z_2 \end{vmatrix} e^{-\frac{1}{2}|z_1|^2} e}$$

$$= \begin{vmatrix} e^{-\frac{1}{2}|z_1|^2} & e^{-\frac{1}{2}|z_2|^2} \\ z_1 e^{-\frac{1}{2}|z_1|^2} & z_2 e^{-\frac{1}{2}|z_2|^2} \end{vmatrix}$$

$$= e^{-\frac{1}{2}|z_1|^2} e^{-\frac{1}{2}|z_2|^2} \begin{vmatrix} 1 & 1 \\ z_1 & z_2 \end{vmatrix}$$

Generalising this to N particles, we get

$$\psi(z_1, z_2 \ldots z_N) = \begin{vmatrix} 1 & 1 & \ldots & 1 \\ z_1 & z_2 & \ldots & z_N \\ z_1^2 & z_2^2 & \ldots & z_N^2 \\ \vdots & & & \\ z_1^N & \ldots & \ldots & z_N^N \end{vmatrix} e^{-\frac{1}{2}\sum_{j=1}^{N}|z_j|^2}$$

So if we have the lowest Landau level with N available states. Then this Slater determinant gives you the N-particle wave-fn which describes the filled Landau level. It is a be written as ~~The inter~~ It can also

$$\psi(z_1 \ldots z_N) = \prod_{i<j} (z_i - z_j)\, e^{-\sum_{j=1}^{N} |z_j|^2}$$

It is ~~eas~~ easy to check this for 2 & 3 particles. This wfn is clearly a.s. under exchange. It also obeys the Pauli exclusion pple. ~~Every photon cannot come close~~ The wfn vanishes if 2 particles come close to each other.

With this background, let me now ~~ge~~ get to the Laughlin wave fn.

The QH~~E~~ expt. when performed at lower temperatures, cleaner samples and lower magnetic fields led to the FQHE $I = \frac{\nu e^2}{h} V$ where ν = fraction such as $1/3$, $1/5$, $2/5$, $3/7$ ...

To explain this (at least fractions of the form $1/2m+1$), Laughlin wrote down his celebrated wfn

$$\psi(z_1 \ldots z_N) = \prod_{i<j} (z_i - z_j)^{2m+1}\, e^{-\sum_{j=1}^{N} |z_j|^2}.$$

To see ~~how~~ what is special about this, you have to first understand that at a~~t~~ fraction like $\nu = 1/3$, there are many degenerate possibilities : if there are

N states, There are many ways of filling 1/3 of the states. Even if N were 9, and we had to fill 1/3 the states, there would $^9C_3$ ways of doing it $= 9 \times 8 \times 7$

So to choose one unique sol^n from that is hard. What Laughlin showed was that in the FQHE, what was important was the Coulomb interaction between electrons. Because of Coulomb repulsion, electrons did not like to come close to each other. So they not only got a simple zero due to Pauli repulsion, they got a triple zero — the wave-fn vanishes very fast if you bring 2 electrons close to one another. There could be other wfns one can write down as well. But it turned out that these wfns were more or less the correct wfns and that is why he got the Nobel Prize a few years ago.

It also turned out that these wfns had interesting props. Excitations over the FQHE gnd state turned out to be neither bosonic nor fermionic. They turned out to be anyonic. And that will be the next and last thing I will discuss in these lectures